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Subset Selection with Whitening and Tikhonov Regularization (SSWT): A Novel and Robust Sensitivity-Based Method for Static Damage Detection Under Load and Measurement Uncertainty

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ABSTRACT

This paper introduces a novel two-stage algorithm for structural damage detection under static loading conditions, explicitly addressing uncertainties in both applied loads and sensor measurements. We rigorously demonstrate that the combined effect of load and sensor errors can be equivalently represented as sensor errors with a specifically defined covariance matrix. The first stage of the proposed method focuses on identifying the location of damaged elements. This is achieved by initially transforming the error covariance matrix into an identity matrix via a linear whitening transformation, effectively converting the ellipsoidal error probability density function (PDF) into a spherical form. Subsequently, the damaged elements are identified by locating those whose displacement sensitivity vectors most effectively span the subspace containing the displacement change vector. The second stage then quantifies the extent of damage using Tikhonov regularization, without requiring iterative model updating. The efficacy of the proposed Subspace Selection with Whitening and Tikhonov Regularization (SSWT) algorithm is rigorously validated through numerical simulations on a truss structure subjected to various damage scenarios and under the influence of uncertainties in both sensors and applied loads. The results unequivocally demonstrate the superior performance of SSWT compared to existing methods in the literature.

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Introduction

Damage detection is one of the branches of structural health monitoring which has recently attracted many scientific efforts. Visual inspection of structures for damage detection is very difficult to be done in many situations. Moreover, there are damage scenarios that are not detectable through inspection. Damage detection techniques have been successfully applied to several real-world problems. Damage is usually simulated by changing in some parameters such as Young's modulus, cross sectional area, moment of inertia and/or boundary conditions, etc. Due to damages, the structural stiffness is reduced; meanwhile, the mass does not change. Recent work widely employs meta-heuristic searching tools for damage detection. For example, Koh et al. [1] detected the damages using accelerations of a shear building under dynamic loads. To this aim, they improved the GA in two ways by embedding two different types of local search algorithms in the GA body. In a study, Koh and Dyke [2] detected damages of a cable stayed bridge by employing the GA for maximizing the response correlation for actual and hypothetically damaged structure, namely, multiple damage location assurance criterion (MDLAC). An application of the micro-genetic algorithm to detect a single crack in a real cracked beam has also been employed by Vakil-Baghmisheh et al.[3]. Begambre and Laier identified the structural damages by using FRFs and employing a hybridized particle swarm optimization with simplex algorithm [4]. Naseralavi et al. [5] improved the real coded genetic algorithm using sensitivity matrices of structural responses with respect to damage severities for diagnosing the damaged elements. Meruane and Heylen [6] also addressed an under-determined damage detection problem. To find the unique damage solution, they penalized the conventional objective function by summation of damage extents. They also hybridized the real-coded genetic algorithm through embedding a local search algorithm in the GA body to improve its performance for damage detection. Yu and Xu [7] employed continuous ACO to detect the damaged elements through minimizing the corresponding objective function. In their study, they used natural frequencies and mode shapes to access the damage.

Many recently proposed methods consist of two stages. In the first stage of these methods, the search space is usually reduced toward damaged elements. In the second stage, the true damage solution is computed by solving the corresponding optimization problem. As an instance, Au et al. [8] used the incomplete mode shapes and natural frequencies for damage detection. To limit the search space, the possible damaged element sites have been approximately identified using the elemental energy quotient difference ratio. Afterwards, a micro-genetic algorithm has been employed for damage detection. Friswell et al. [9] first used a discrete version of the GA for estimating the damage state. After that, they employed a sensitivity-based approach to achieve the accurate result. Guo and Li [10] proposed a two-stage method. In their first stage, the damage sites are localized by using evidence theory from both natural frequencies and mode shapes. In the second stage, the damage extents are determined by using genetic algorithm. Naseralavi et al. [11] presented a method called subset solving algorithm for damage detection of structures. In the first stage, the structural elements are ordered based on their damage probability into a vector referred to as elements damage probability ordering vector (EDPOV). In the second stage, rather small subsets of EDPOV elements are judiciously selected to form a nonlinear system of equations, which are subsequently solved to detect potential damages. He and Hwang [12] proposed a two-stage method; in its first stage, the search space is reduced by grey relation analysis. In the next stage, a real-parameter genetic algorithm is combined with simulated and adaptive mechanisms for finding the actual damages.

Static damage identification methods are usually simpler than the dynamic ones, since the static equilibrium equation is only relevant to the stiffness properties of structures. Moreover, the equipment's for static testing are comparatively cheaper [12]. One advantage of using static responses is that they are more sensitive to damages than natural frequencies and mode shapes[13]. Conversely, static responses are less popular in the case of highly stiff structures due to implementation problems. For structures under static loads, the deformations and strains can be measured with high accuracy. Combinations of static response with other types of responses are utilized for damage diagnosis in the previous work. Cao et al. [14] identified damage in cantilever beams by using the sensitivity of fundamental mode shape and static deflection. Wang et al. [15], at first, defined a new simple function constructed by frequencies and a static response called damage signature. They localized the structural damages by matching damage signature in two states of a real damaged structure and an analytical model. Then, they iterated a quadratic programming until the damages were detected. Static responses are also merely employed for damage detection for many times [16-22]. Hjelmstad and Shin [20] detected the damage of a planar bowstring truss structure using the static response. To this aim, they detected the damages through minimizing the squared model error while employing an adaptive parameter grouping method. Bakhtiari-Nejad et al. [21] introduced a damage identification scheme based on the quadratic programming method. They also proposed some methods for optimizing load and sensor placements. Wu and Law [22] localized damages of plates by a variation of curvatures under uniform static load.

However, a significant challenge in static damage detection lies in effectively handling uncertainties arising from noisy sensor measurements and inaccuracies in applied loads. These uncertainties can significantly affect the accuracy and reliability of damage identification algorithms. To address this critical limitation, this paper introduces a novel two-stage sensitivity-based method, termed Subspace Selection with Whitening and Tikhonov Regularization (SSWT), specifically designed for robust static damage detection in large-scale structures under such uncertainties. The first stage of SSWT employs a whitening transformation to mitigate the effects of noise by converting the error covariance structure into an identity matrix. This ensures that the subsequent subset selection process, which identifies potential damage locations based on the sensitivity of displacement vectors, is not biased by the inherent structure of the noise. The second stage then utilizes Tikhonov regularization to accurately quantify the damage extents, providing a stable and reliable solution even in the presence of significant uncertainties. The proposed SSWT method is particularly well-suited for large-scale structures, where the number of potential damage locations can be substantial. The efficacy of SSWT is demonstrated through rigorous numerical studies on large-scale steel truss structures. This work significantly advances the field by providing a robust and accurate methodology for static damage detection in the presence of realistic uncertainties.

The remainder of this paper is organized as follows: Section 2 provides the theoretical background on damage detection using sensitivity analysis in a static approach, followed by a detailed description of the proposed SSWT methodology in Section 3. Section 4 elaborates on the handling of errors in forces and measurements, including the whitening transformation. The proposed method is presented in Section 5. Some other sensitivity-based methods for comparison with the first stage of the proposed method for recognizing the damage candidates are recognized are given in Section 6. The numerical case study and results are presented in Section 7, and finally, Section 8 concludes the paper with a summary of the key findings and potential future research directions.

Damage detection using sensitivity analysis in static approach

Damage within a structure leads to duction in the stiffness of the affected elements. Consequently, when subjected to static loads, a damaged structure exhibits increased displacements compared to its healthy state. This increase in displacements under a defined load serves as a primary indicator of the presence of damage. Furthermore, the magnitude and distribution of these displacement changes can provide insights into the location and extent of the damage. The static equilibrium of the structure is described by:

$$\mathrm{KU} = \mathrm{F} \tag{1}$$

where **K** is the stiffness matrix, **U** is the displacements vector, and **F** is the applied loads vector. For damage identification, one has to find the set of damage variables in a way that the analytical displacements of the structure would best fit the measured ones. Damage detection problems can be expressed mathematically as below [5]:

$$\mathrm{U}_d = \mathrm{U}(\mathrm{X}) \Rightarrow \mathrm{X} = ? \tag{2}$$

where $\mathrm{X} = (x_1,x_2,...x_n)^{\mathrm{T}}$ is called the damage vector, and $0 \leq x_i \leq 1$ is the damage extent (damage ratio) of the *i*th element where $x_i = 0$ and $x_i = 1$ indicate the intact and completely damaged states, respectively; and *n* is the number of structural elements. $\mathrm{U}(\mathrm{X}) = (\mathrm{u}_1(\mathrm{X}),\mathrm{u}_2(\mathrm{X}),...\mathrm{u}_m(\mathrm{X}))^{\mathrm{T}}$ is the vector of *m* responses of hypothetically damaged structure with damage state **X** that can be evaluated from the analytical model and $\mathrm{U}_{\mathrm{d}}=(\mathrm{u}_{\mathrm{d},1},\mathrm{u}_{\mathrm{d},2},...\mathrm{u}_{\mathrm{d,m}})^{\mathrm{T}}$ is the vector of *m* structural responses of the existing damaged structure. Generally speaking, the nonlinear system of equations presented in Eq. (2) can be solved by using heuristic optimization methods, sensitivity-based methods, or a combination of both [5; 6]. This equation can be estimated by using the first order approximation as follows [5]:

$$\mathrm{U}_{\mathrm{d}}=\mathrm{U}(\mathrm{X})=\mathrm{U}_{\mathrm{h}}+\frac{\partial \mathrm{U}}{\partial \mathrm{X}}\Delta\mathrm{X}+... \Rightarrow \overbrace{\mathrm{U}_{\mathrm{d}}-\mathrm{U}_{\mathrm{h}}}^{\Delta \mathrm{U}} \simeq \overbrace{\frac{\partial \mathrm{U}}{\partial \mathrm{X}}}^{\mathrm{S}}(\mathrm{X}_{\mathrm{d}}-\overbrace{\mathrm{X}_{\mathrm{h}}}^{0_{n\times 1}}) \tag{3}$$
$$\therefore \mathrm{S}_{m\times n}.\mathrm{X}_{n\times 1}=\Delta\mathrm{U}_{m\times 1}$$

where U_h is the structural displacement vector of the healthy structure, $\Delta\mathbf{X}$ is the damage change vector caused by actual damage, $\mathrm{S} = \partial\mathrm{U}/\partial\mathrm{X}$ is the sensitivity matrix, and $\mathrm{X}_{\mathrm{d}}=\left(\mathrm{x}_{\mathrm{d},1},\mathrm{x}_{\mathrm{d},2},...,\mathrm{x}_{\mathrm{d},n}\right)^{\mathrm{T}}$ is the actual damage vector. Notice that $\mathrm{U}(0) = \mathrm{U}_h$ and $\mathrm{U}(\mathrm{X}_d) = \mathrm{U}_d$. $\Delta\mathrm{U} = \mathrm{U}_d - \mathrm{U}_h$ is the vector of the structural response change due to the damage. We denote the *i*th column of **S** as S_{i}, and hence $\mathrm{S}=\left[\mathrm{S}_1,\mathrm{S}_2...\mathrm{S}_{\mathrm{n}}\right]$.

Methodology

Due to the fact that in static response the number of measurements is usually less than the number of structural members, so the equation ΔU=S.X is under-determined ($n > m$), and hence has infinite solutions. Mathematically speaking, there is an $(n -$

m)-dimensional subspace in R^n that its points are solutions of ΔU=S.X. To find the true unique damage solution within the subspace, it should be noted that in the damaged structures most of the structural elements are still intact and, therefore, the true solution has high sparsity. In the first stage of the algorithm, we restrict the damage candidates by considering some of the elements healthy and eliminating them from the design variables. Since $\Delta U \simeq S.X_d = \sum_{i=1}^{n} x_{d,i} S_i$, and the damage ratio of the undamaged elements are zero, thus we can write $\Delta U \simeq \sum_{i \in J} x_{d,i} S_i$, where J is the set of damaged elements. This means ΔU lies approximately in the span of such S_i's corresponding to the damaged elements, i.e., ΔU lies approximately in $\text{span}\{S_i : i \in J\}$. In other words, ΔUlies approximately in the subspace spanned by S_i's of the damaged elements. Therefore, it is expected that the component of ΔU orthogonal to $\text{span}\{S_i : i \in J\}$ is smaller in magnitude than orthogonal component of ΔU with respect to other same dimensional subspaces spanned by other combinations of S_i's. Here, the same dimension condition for such a comparison is imposed due to the fact that by enlarging a subspace through adding new dimensions to it, the orthogonal component of a certain vector to that subspace will get shortened. The component of ΔU along S_i is approximately equal to the damage ratio of the *i*th element. Let us review the above discussions with an example for better understanding. Consider a structure in which the elements 4, 6, and 8 are damaged. Therefore, ΔU lies approximately in the subspace span(S_4,S_6,S_8), i.e., the orthogonal vector $\Delta U_{\perp \text{span}(S_4,S_6,S_8)}$ which is denoted by *f* in the figure has a very small magnitude. Meanwhile, the orthogonal component of ΔU to any other 3-dimensional subspaces spanned by any other 3-combinations of S_i's is not generally that small.

As aforementioned, to find the damaged elements, one might excessively search for such S_i's so that ΔUbest lies in the subspace spanned by them, and consequently take the corresponding elements as damaged ones. To ease the explanation two definitions should be considered as follows:

Definition 1:

Consider matrix $S_{n\times m}$ and J⊂{1,2,...m}, then $S(J)$ is defended as submatrix of S having the columns of S corresponding to elements of J. For example if J={3,7,12} then $S(J)$is matrix having three columns consisting of the 3rd , 7th and 12th columns of matrix S, that is S(J)=[S_3,S_7,S_{12}].

Definition 2:

The expression $\text{span}(S(I))$ is considered as $\text{span}\{S_i : i \in I\}$which is the subspace spanning by the columns of $S(J)$.

The vector ΔU can be decomposed to the component lies in $\text{span}(S(I))$ and the component perpendicular to the subspace, i.e. $\Delta U = \text{proj}_{\text{span}(S(I))}\Delta U + \Delta U_{\perp \text{span}(S(I))}$. Therefore the expression $\left\| \Delta U\text{-proj}_{\text{span}(S(I))}\Delta U \right\|$ represents the magnitude of component of ΔU which is perpendicular to the subspace $\text{span}(S(I))$. Therefore, the problem of damage detection can be considered as the following optimization problem:

Objective function (I) to be minimized: $\left\| \Delta U\text{-proj}_{\text{span}(S(I))}\Delta U \right\|$ Subject to:

$$I \subset \{1,2,...m\} \ \& \ |I| = l \tag{4}$$

in which the cost function, F(I), should be minimized, where $\|.\|$ and $|.|$ denote Euclidean norm of a vector and the number of members of a set, respectively. *l* is the maximum number of damaged elements. Noteworthy, *l* value makes an upper bound for the number of damages in which the algorithm is valid. As an instance, in the case of setting *l* to 4, the proposed method is applicable only for the cases up to 4 damaged elements.

Error in forces and measurements

Generally, in static approaches the noise exists in both applied loads and measurements. Here we are to see how to combine the effect of these noises. First, we assume that the noise distribution for both loads and measurements is multivariate Gaussian. When we think that the load vector F is applied, due to noise of load the load is equal to $F_1=(I+\xi_1\times \mathrm{diag}(\varepsilon_1)).F$, where ξ_1 is the noise level for loads and ε_1 is a random vector from Gaussian noise with zero mean and identical covariance, i.e. $\varepsilon_1 \sim N(0_{n\times 1}, I_{n\times n})$. The structural displacements in the case of loads with noise would be equal to:

$$U_1=K^{-1}F_1=K^{-1}\times(I+\xi_1\times \mathrm{diag}(\varepsilon_1)).F=K^{-1}F+\xi_1\times K^{-1}.\mathrm{diag}(\varepsilon_1).F \tag{5}$$

The second term in the above equation, $\xi_1\times K^{-1}.\mathrm{diag}(\varepsilon_1).F$, is the effect of error in loads. This is the exact displacements in the presence of load errors. However, duo to noise of sensors the vector $U_2=(I+\xi_2\times \mathrm{diag}(\varepsilon_2)).U_1$ is measured, where ξ_2 is the noise level for sensors and ε_2 is a random vector from standard Gaussian noise, i.e. $\varepsilon_2 \sim N(0_{n\times 1}, I_{n\times n})$. By substituting Eq. (3) into the expression $U_2=(I+\xi_2\times \mathrm{diag}(\varepsilon_2)).U_1$, we will have:

$$\begin{aligned} &U_2=(I+\xi_2\times \mathrm{diag}(\varepsilon_2))\times\left(K^{-1}F+\xi_1\times K^{-1}.\mathrm{diag}(\varepsilon_1).F\right)= \\ &K^{-1}F+\xi_1\times K^{-1}.\mathrm{diag}(\varepsilon_1).F+\xi_2\times \mathrm{diag}(\varepsilon_2)K^{-1}F+ \\ &\xi_1\xi_2 \mathrm{diag}(\varepsilon_2)K^{-1}.\mathrm{diag}(\varepsilon_1).F \end{aligned} \tag{6}$$

Since both ξ_1 and ξ_2 are generally small values, therefore the forth term in Eq. (2) which consists of $\xi_1.\xi_2$ is a very small value and thus can be neglected. Hence, Eq. (6) can be written as:

$U_2=K^{-1}F+\xi_1\times K^{-1}.\mathrm{diag}(\varepsilon_1).F+\xi_2\times \mathrm{diag}(\varepsilon_2).K^{-1}F$. On the other hand, we have: $\mathrm{diag}(\varepsilon_1).F=\mathrm{diag}(F).\varepsilon_1$ and $\mathrm{diag}(\varepsilon_2).U=\mathrm{diag}(U).\varepsilon_2$. As a consequence, Eq. (6) can be rewritten as:

$$U_2=U+\xi_1\times K^{-1}.\mathrm{diag}(F).\varepsilon_1+\xi_2\times \mathrm{diag}(U).\varepsilon_2$$

or equivalently:

$$U_2=U+M.\varepsilon_1+N.\varepsilon_2 \tag{7}$$

where $M = \xi_1 \times K^{-1} \times \mathrm{diag}(F)$ and $N = \xi_2 \times \mathrm{diag}(U)$.

In Eq. (7) the terms $M.\varepsilon_1$ and $N.\varepsilon_2$ are the effects of error in loads and measurements, respectively. Therefore, distribution of measured displacements in the presence of errors is a Gaussian distribution as:

$$\varepsilon \sim N(0, \Sigma=MM^T+NN^T) \tag{8}$$

This error is a Gaussian white noise with zero mean and the covariance $\Sigma = MM^T + NN^T$. Thus, the expression $S.X=\Delta U$in the presence of the noises is transformed to: $S.X=\Delta U+\varepsilon$.

Whitening of total noise:

Geometrically, a Gaussian noise can be considered as a cloud, where selection probability of each point is proportional to the density of cloud at that point. The equal density contours of the cloud are proportional ellipsoids with the same centres and the same directions in principal axes. The principal axes of the ellipsoids are the eigenvectors of covariance matrix. Also, the lengths of axes are proportional to square root of the eigenvalues corresponding to the eigenvectors. Whitening is a linear transformation which makes the covariance an identity matrix. As a result, the ellipsoid cloud of noise is transformed into a spherical cloud. Ellipsoid shape of PDF causes the discriminate effect of errors in various directions.

Here we are to find whitening transformation matrix. Consider eigen-decomposition of the covariance as: $\Sigma=\Phi\Lambda\Phi^{T}$, where Φ is the eigenvectors of covariance matrix and Λ is a diagonal matrix with eigenvalues in the main diagonal. The expression $\Phi\Lambda\Phi^{T}$ can be written as: $(\Phi\Lambda^{1/2})(\Phi\Lambda^{1/2})^{T}$. Thus, through pre-multiplying by the matrix $T=(\Phi\Lambda^{1/2})^{-1}$, the covariance would change into the identity matrix: $(\Phi\Lambda^{1/2})^{-1}(\Phi\Lambda^{1/2})(\Phi\Lambda^{1/2})^{T}(\Phi\Lambda^{1/2})^{-T}=I$, and as a result the ellipsoids contours are transformed to spheres. By pre-multiplying the sides of $S.X=\Delta U+\varepsilon$ by matrix **T**, we would have:

$$(T.S).X=T.\Delta U+T.\varepsilon \tag{9}$$

By denoting T.S, $T.\Delta U$ and $T.\varepsilon$ by S^{*}, ΔU^{*} and ε^{*}, respectively, we can rewrite Eq. (9) as:

$$S^{*}.X=\Delta U^{*}+\varepsilon^{*} \tag{10}$$

where ε^{*} is a Gaussian with spherical distribution. Thus, matrix **T** changes a linear system of equations with ellipsoid Gaussian noise to another system of equations with hyper-sphere Gaussian noise. Matrix **T** is called whitening matrix.

Probability of damage existence:

Generally, the significant difference between the analytical healthy model and measured data is considered as the sign of damage presence in structure. However, it possible that the structure is healthy, and the difference between the measured response and the responses of healthy structure is only due to noise effects. Here, we address that what is the probability that such a difference occurred while the structure is healthy.

Instead of using $S.X=\Delta U+\varepsilon$, the equation $S^{*}.X=\Delta U^{*}+\varepsilon^{*}$ is considered, where $\varepsilon^{*}\sim N(0,I)$. Probability of damage existence is equal to:. $P(\|\varepsilon^{*}\|<a)=\int_0^a f_{\chi(m)}(x).dx$

Whenever $P(\|\varepsilon^{*}\|<a)=\int_0^a f_{\chi(m)}(x).dx>0.95$ the structure is considered damaged. The probability that the elements of set *I* are the damaged elements of the structure only, is equal to:

$$P(\|\varepsilon^{*}\|<a)=\int_0^a f_{\chi(m\text{-}i)}(x).dx \tag{11}$$

If the elements of set *I* are the damaged elements, then $\|\Delta U^{*}-\mathrm{proj}_{\mathrm{span}(S^{*}(I))}\Delta U^{*}\|$ would be a very small value in comparison with other sets with the same number of elements [7]. The amount of $\varepsilon=\|\Delta U^{*}-\mathrm{proj}_{\mathrm{span}(S^{*}(I))}\Delta U^{*}\|$ mostly depends the amount of errors in loads and sensors. The probability $P(\|\varepsilon^*\|<a)=\int_0^a f_{\chi(m-i)}(x).\,dx>0.95$ it could be considered as the stop criterion, because it is a sign of presence of other elements rather than elements of set *I*. The value of CDF of the chi-distribution with $m-i$degree of freedom at the point $\|\varepsilon\|$ can also be denoted by $CDF(\|\Delta U^*\|;\chi_{m-i})<0.95$. Consequently, for finding damaged elements if the stop criterion does not meet, another search among sets with one more element should be performed. Increment by one-unit among sets are executed until the stop criterion get met. in the case that I is subset of the damaged elements of structure, the amount of $\|\Delta U^* - \mathrm{proj}_{\mathrm{span}(S^*(I))}\Delta U^*\|$ can be indicated whether I consists of all damaged elements or not. Consequently if $CDF(\|\Delta U^* - \mathrm{proj}_{\mathrm{span}(S^*(I))}\Delta U^*\|;\chi_{m-1})<0.95$, *I* consists of all damaged elements and otherwise there are some damaged elements which are not present in *I*. Noteworthy without whitening of vector of errors, the chi-distribution could not be applicable for determination of stop criterion.

Hence whenever $P(\|\varepsilon^*\|<a)=\int_0^a f_{\chi(m-i)}(x).\,dx>0.95$, it is considered that there are *I* damaged elements in the structure; otherwise search among sets with $i+1$ damaged elements is started. We assume that the structure is healthy (null hypothesis) and thus $\Delta U^*=0$ and consequently what is measured by sensors are merely errors with the distribution$\varepsilon^*\sim N(0,I)$. Now a question arises: what is the probability that the structure is healthy. The sum of square of some terms with standard multi-variate normal distribution is chi-square. With the same reasoning, among ℓmember subsets of elements $J\subset\{1,2,\dots m\}$ the amount of out of subspace component in subset selection with whitening, $\|\Delta U^* \text{-} \mathrm{proj}_{\mathrm{span}(S^*(I))}\Delta U^*\|$, can be considered whether it induced due to random errors or there are more than ℓ damaged elements. and there are other damaged elements that should be found.

Subset Selection with Whitening and Tikhonov Regularization (SSWT)

The present method consists of two stages for damage detection of structures without updating the sensitivity matrix. In the first stage the damage candidates of structure are determined. By means of a well-established whitening process, the method becomes resistant to randomness effect of errors in sensors and applied loads. As described previously, this is done by pre-multiplication of S, ΔU by $T=(\Phi\Lambda^{1/2})^{-1}$ where$\Sigma=\Phi\Lambda\Phi^T$to obtain $S^*=T.S$, $\Delta U^*=T.\Delta U$. Whitening process makes the ellipsoid cloud of errors to be added to ΔU spherical. That is, after whitening the covariance matrix of multivariate normal distribution becomes the identity matrix, I. This also helps to find the number of damaged elements through hypothesis test about the magnitude of $\|\Delta U^* - \mathrm{proj}_{\mathrm{span}(S^*(I))}\Delta U^*\|$. If this magnitude is larger than $P(\|\varepsilon^*\|<a)=\int_0^a f_{\chi(m-i)}(x).\,dx>0.95$ it means that the amount of $\|\Delta U^* - \mathrm{proj}_{\mathrm{span}(S^*(I))}\Delta U^*\|$ is unlikely to be originated form the errors only and there are other elements rather than the element of I which are within the damaged elements of the structure. Next in the second stage the damage extents of the determined damaged elements are derived.

Stage one: selection of damaged elements:

In the first stage of the proposed algorithm using the presented methodology an algorithm as follows. The algorithm is based on the best subspace generated from the sensitivity of response.

<u>**Subset selection for damage candidates**</u>

$l \leftarrow 1,\ flag \leftarrow 0$

While $flag = 0$ **do**

Find the l-element subset $I \subset \{1, 2, \ldots, n\}$ producing the smallest $\left\| \Delta\mathbf{U}^* - \mathrm{proj}_{\mathrm{span}(\mathbf{S}^*(I))} \Delta\mathbf{U}^* \right\|$ and consider the set as J :

$J \leftarrow \arg\min_I (\left\| \Delta\mathbf{U}^* - \mathrm{proj}_{\mathrm{span}(\mathbf{S}^*(I))} \Delta\mathbf{U}^* \right\|)$ where I is a subset of $\{1, 2, \ldots, n\}$ having l elements, i.e. $I \subset \{1, 2, \ldots, n\}$ & $|I| = l$.

If $\mathrm{CDF}(\min(A_l); \chi_{m-l+1}) < 0.95$,

then:

$l \leftarrow l + 1$

Otherwise:

$flag \leftarrow 1$

End if

End while

Report elements of set J as set of all damage candidates

Figure 1. Pseudo-code of the proposed method (subset selection with whitening)

Stage two: determining damage severities:

Damage extent of the damaged elements is conventionally calculated as:

$$[S(J)]\{X\} = \{\Delta U\} \Rightarrow \{X\} = [S(J)]^{+}\{\Delta U\} \tag{12}$$

in which "+" represents Moore-Penrose pseudo inverse.

However, in the case $\{\Delta U\}$ is contaminated with Gaussian error, Tikhonov regularization can be employed to get better results. Tikhonov regularization performs by scaling Euclidean space in the direction of diameters of noise ellipsoid. This results in downsizing the large diameters of the ellipsoid improves the solution (advantageous effect), meanwhile the centre of the ellipsoid unwillingly moves (disadvantageous effect) which worsens the solution. Tikhonov regularization affects by these two actions for solution of linear system of equations. Tikhonov regularization is applied as follows:

$$\begin{bmatrix} S(J) \\ \lambda I \end{bmatrix} \{X\} = \begin{Bmatrix} \Delta U \\ 0 \end{Bmatrix} \Rightarrow \{X\} = \begin{bmatrix} S(J) \\ \lambda I \end{bmatrix}^{+} \begin{Bmatrix} \Delta U \\ 0 \end{Bmatrix} \tag{13}$$

The parameterλin Tikhonov regularization adjusts the magnitude of filtering. The amount of the parameter λ in Tikhonov regularization should be selected that mush the advantageous effect prevails the disadvantageous effect. Considering $\mu = (\lambda - \sigma_n)/(\sigma_1 - \sigma_n)$ where $0 \leq \sigma_n \leq \ldots \leq \sigma_2 \leq \sigma_1$ are the singular values of the covariance Σ then $\mu = 0.9$ is a beneficial value resulting in better damage solution based on our numerical results. Error diameters corresponding to small sizes of $\sigma_i$shoots the solution to unfeasible solutions and its values depend on the amount of noise in the data. Fig. 2 represents the summary of the proposed method in two stages.

Stage 1: Detecting damage elements J

Step 1) Whitening noise with the covariance Σ

$$\mathbf{M} = \xi_1 \times \mathbf{K}^{-1} \times \mathrm{diag}(\mathbf{F}),\ \ \mathbf{N} = \xi_2 \times \mathrm{diag}(\mathbf{U})$$
$$\boldsymbol{\Sigma} = \mathbf{M}\mathbf{M}^T + \mathbf{N}\mathbf{N}^T$$
$$\boldsymbol{\Sigma} = \boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^T$$
$$\mathbf{T} = (\boldsymbol{\Phi}\boldsymbol{\Lambda}^{1/2})^{-1}$$
$$\Delta\mathbf{U}^* \leftarrow \mathbf{T}.\Delta\mathbf{U}$$
$$\mathbf{S}^* \leftarrow \mathbf{T}.\mathbf{S}$$

Step 2)

$i \leftarrow 1$

while for each set I with i elements

$\quad \mathrm{CDF}(\left\|\Delta\mathbf{U}^* - \mathrm{proj}_{\mathrm{span}(\mathbf{S}^*(I))}\Delta\mathbf{U}^*\right\|; \chi_{m-i}) > 0.95,$ *then* :

$\quad i \leftarrow i+1$

else

$\quad J \leftarrow \arg\min_I (\left\|\Delta\mathbf{U}^* - \mathrm{proj}_{\mathrm{span}(\mathbf{S}^*(I))}\Delta\mathbf{U}^*\right\|)$

end

Stage 2:

Determining damage extents of J using Tikhonov regularization

$$\{\mathbf{X}\} = \begin{bmatrix} \mathbf{S}(J) \\ \lambda\mathbf{I} \end{bmatrix}^+ \begin{Bmatrix} \Delta\mathbf{U} \\ \mathbf{0} \end{Bmatrix}$$

Figure 2. Flowchart of the proposed method subset selection with whitening and Tikhonov regularization (SSWT)

Some methods for comparison with the first stage of SSWT

To show the efficiency of the first stage of the proposed method for determining the damage candidates, it should be compared with some others methods. All methods are similar in using the sensitivity matrix without updating the model. For verification the influence of whitening process, subspace selection without whitening is also considered. The subspace selection without whitening shows the proficiency of whitening process in stabilizing the subspace selection method. Moreover, the well-known forward selection method is regarded as another method for compassion [7].

Pseudo-code

Consider the number of damage candidates as *nd*

$i \leftarrow 1,\ \Delta\mathbf{U}_1 \leftarrow \Delta\mathbf{U},\ J \leftarrow \{\ \}$

While $i \le nd$ **do**

$\quad L_i \leftarrow \arg\max_j \left\|\mathrm{proj}_{S_j}\Delta\mathbf{U}\right\|;\ \ j = 1, 2\ldots, n$

$\quad \Delta\mathbf{U}_{i+1} \leftarrow \Delta\mathbf{U}_i - \mathrm{proj}_{\mathbf{S}_{L_i}}\Delta\mathbf{U}_i$

$\quad \mathbf{S}_{L_i} \leftarrow \mathbf{S}_{L_i} - \mathrm{proj}_{\mathbf{S}_{L_i}}\Delta\mathbf{U}_i;\ \ i = 1, 2\ldots, n$

$\quad i \leftarrow i+1$

$\quad J \leftarrow J \cup \{L_i\}$

End while

Report elements of set J as set of all damage candidates

Figure 3. Pseudo-code of forward selection without whitening

Pseudo-code
Consider the number of damage candidates as *nd*

$i \leftarrow 1,\ \Delta\mathbf{U}_1 \leftarrow \Delta\mathbf{U},\ J \leftarrow \{\ \}$

While $i \le nd$ **do**

$L_i \leftarrow \arg\max_j \left\| \mathrm{proj}_{S_j} \Delta\mathbf{U}^* \right\|;\ j = 1, 2..., n$

$\Delta\mathbf{U}^*_{i+1} \leftarrow \Delta\mathbf{U}^*_i - \mathrm{proj}_{\mathbf{S}^*_{L_i}} \Delta\mathbf{U}^*_i$

$\mathbf{S}^*_{L_i} \leftarrow \mathbf{S}^*_{L_i} - \mathrm{proj}_{\mathbf{S}^*_{L_i}} \Delta\mathbf{U}^*_i;\ i = 1, 2..., n$

$i \leftarrow i + 1$

$J \leftarrow J \cup \{L_i\}$

End while

Report elements of set J as set of all damage candidates

Figure 4. Pseudo-code of forward selection with whitening

In each step of forward selection, the member corresponding to the best sensitivity vector having the most component in the direction of response change is chosen as the next damage candidate. The well-known forward selection method is considered in both cases with and without whitening of errors. In these methods *nd* is the number of damage candidates that should be preassigned before the start of the methods. The pseudo codes of the considered methods are presented in Figs. 2-5.

Subset selection for damage candidates

$l \leftarrow 1,\ l \leftarrow nd$

Find the *l*-element subset $I \subset \{1, 2, ..., n\}$ producing the smallest

$\left\| \Delta\mathbf{U}^* - \mathrm{proj}_{\mathrm{span}(\mathbf{S}^*(I))} \Delta\mathbf{U}^* \right\|$ and consider the set as J :

$J \leftarrow \arg\min_I \left(\left\| \Delta\mathbf{U}^* - \mathrm{proj}_{\mathrm{span}(\mathbf{S}^*(I))} \Delta\mathbf{U}^* \right\| \right)$ where I is a subset of

$\{1, 2, ..., n\}$ having l elements, i.e. $I \subset \{1, 2, ..., n\} \,\&\, |I| = l$.

Report elements of set J as all damage candidates

Figure 5. Pseudo-code of subset selection without whitening

Case study

The efficiency of the proposed method is investigated by a steel truss having the configuration and geometry depicted in Fig. 6. This structure previously studied by Bakhtiarinejad et al. [21]. Young's modulus for steel materials is considered to be 2.1×10^6 kg/cm^2. To identify the damaged elements, two load cases with three forces are considered as shown in Fig. 7 a and b. The corresponding sensors for each load case are respectively represented in Fig. 8 a and b. The six damage scenarios given in Table 1 are to be studied. In the table the damaged elements and the amount of damage severities are given. The damage is simulated by a reduction in the Young's modulus of the damaged elements with the amount of damage severity. For errors in measurement of sensors (*N*) and the amount of applied loads (*LE*) are both considered to by normally distributed.

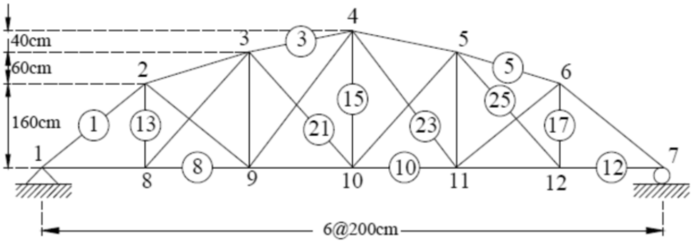


Figure 6. A Planar Truss structure having 25 elements

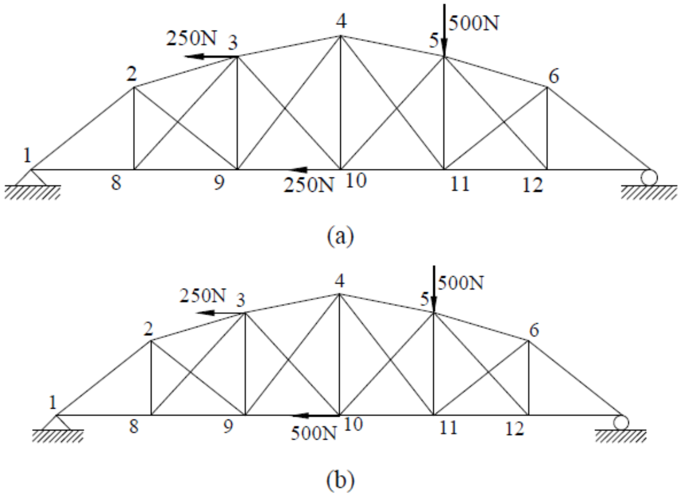


Figure 7. Applied loads in (a) load case 1 (b) load case 2

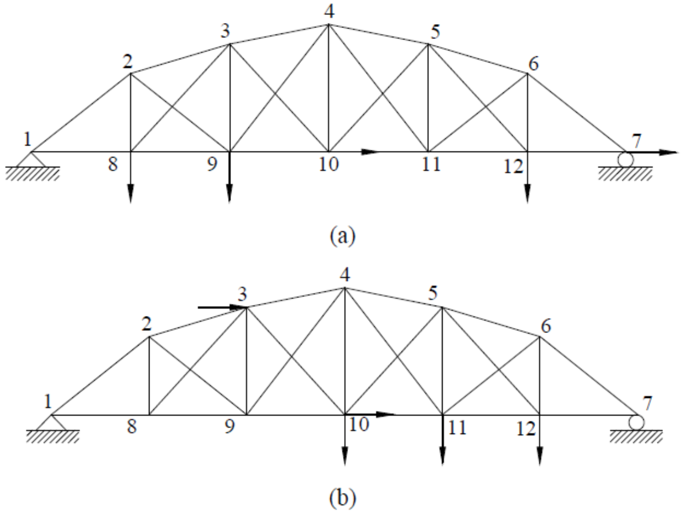


Figure 8. Measured displacements in (a) load case 1 (b) load case 2

In order to investigate the influence of errors in the shape of PDF of displacements, the structure is first considered to be healthy, and then the error in sensors and applied loads is randomly simulated one thousand times. The PDF of errors in sensors and applied loads are individually considered to be univariate Gaussian with the variance of 0.01. The value measured displacements in degrees of freedoms one versus six and degrees of freedom two versus three are respectively depicted in Fig. 9 and 10 in both cases with and without whitening. As it can be seen from the figures before application of whitening the shape of PDF is ellipsoid with far difference in the size of diameters. After applying whitening the PDF become spherical (Figs. 9 and 10 b). The existence of errors with ellipsoid PDF causes the measured displacements shoot in the direction of large diameters of ellipsoid. This shooting results in the inefficiency of the subspace selection method because shooting causes undesirable out subspace component in the response change. Whitening makes the PDF of error become spherical and shooting of response error has the same likelihood in various directions. Consequently, application of whitening helps the algorithm to resist against randomness of errors, since it prevents the discriminate effect of error in various directions.

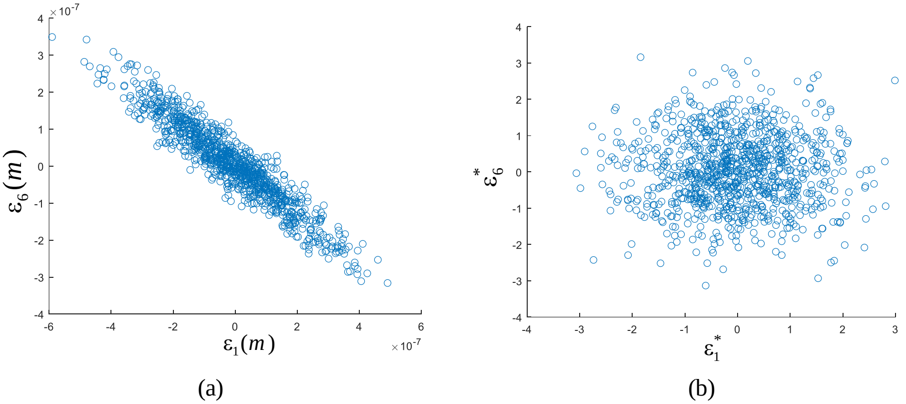
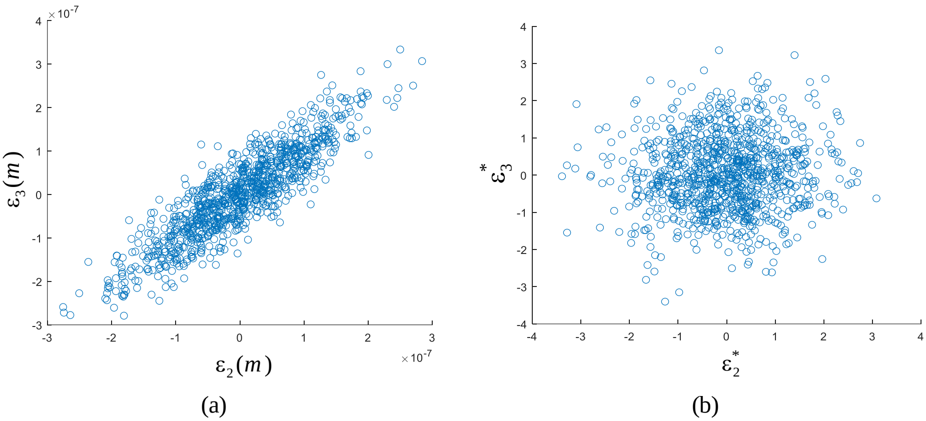


Figure 9. Correlation of errors in degrees of freedom 1 and 6 in 1000 randomly generated errors in loads and measurements (a) without whitening (b) with whitening



Figute 10. Correlation of errors in degrees of freedom 2 and 3 in 1000 randomly generated errors in loads and measurements (a) without whitening (b) with whitening

Table 1. Damage scenarios

Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5	Scenario 6
Element 8 30%	Element 11 20%	Element 2 20%	Element 8 20%	Element 1 25 %	Element 3 10%
Element 14 20%	Element 20 20%	Element 9 20%	Element 14 20%	Element 11 35 %	Element 23 40%
Element 20 20%		Element 24 30%	Element 19 20%	Element 17 20%	

The damaged elements and damage extents are given in Table 1 for six damage scenarios. The first stage of the proposed method consists of subspace selection with whitening. For investigation of the first stage of the proposed method, the subspace selection and forward selection in the both cases with and without whitening are compared. In damage scenario one, the elements 8, 14 and 20 are considered to be damaged. For this damage scenario, the amount of errors in applied loads and measurement of sensors are both considered to be 0.3%. The results are given in Table 2 in separate steps. As it can be seen all method distinguish element 8 as the first damaged element at the first step. In the second step the methods of forward selection and subspace selection in the case with whitening choice element 14 and in the case without whitening choice 20 as damaged element. Therefore, all methods have been successful in diagnose of damaged elements in the first two steps. However, both forward selection and subspace selection method fails to detect the third damaged element in the case of without whitening in step three. Hence, whitening of errors in the both forward selection and subspace selection methods has been successful results and therefore whitening is absolutely beneficial.

At the start of SSWT, the combination of the whitening of errors and subset selection method makes it clear that if the structure is healthy or damaged. Additionally, in the case the structure is diagnosed to be damaged, at each step of the method it becomes clear that if there are more damaged elements to be found or not. The magnitude of $\mathrm{CDF}(\|\Delta\mathrm{U}\|;\chi_{m-i})$ is given in each step in the column related to SSWT method (subspace selection + whitening) in Table 2. Whenever $\mathrm{CDF}(\|\Delta\mathrm{U}\|;\chi_{m-i})$comes lower than 0.95 if reveals there is no extra damaged elements and all damaged elements are already has been found. As it is observed, in the third step it becomes

SSWT truly distinguished the structure is not healthy. Then after the damaged element 8 is recognized the probability there are more damaged elements to be found.

Therefore, the elements 8, 20 and 14 are finally diagnosed as the damaged elements at the end of the first stage of SSWT. The damage extents of the damaged elements should be identified in the second stage of the method. To obtain the optimal value for the regularization parameter the norm of error in damage solution ,$\|X-X_d\|$, versus the regularization parameter $\mu$is depicted in Fig. 11. As it is observed the value of 0.9 for μ which is herein considered for SSWT is near the optimal value. To verify the influence of regularization the damage solution in both cases with and without Tikhonov regularization are calculated. The damage solution results are shown in Fig. 12. It can be observed that the damage solution in the case with Tikhonov regularization is much nearer to the exact damage solution, and hence Tikhonov regularization with $\mu = 0.9$ improves the damage solution.

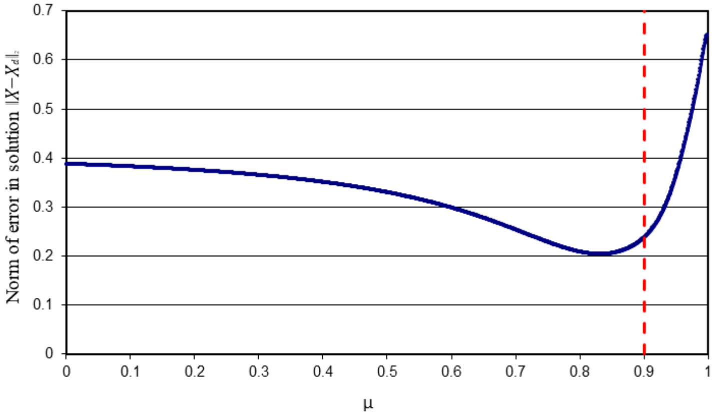


Fig. 11. Norm of error in solution versus μ for damage scenario 1 and N=0.3%, LE=0.3%

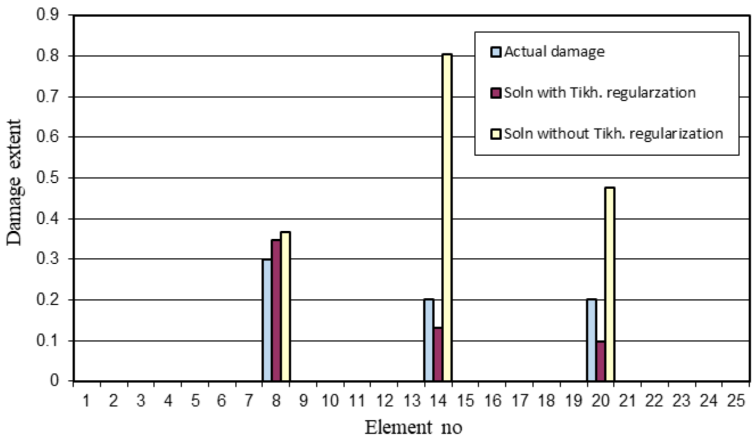


Fig. 12. Damage detection results in the cases of employed and unemployed Tikhonov regularization for damage condition 1

Table 2. Comparison of subspace selection and forward selection in the cases of with and without whitening for damage scenario 1 and N=0.3%, LE=0.3%

Damage scenario 1, N=0.3% LE=0.3% $\frac{\lambda_{max}}{\lambda_{min}}$=35.06	**Subspace selection + whitening CDF(‖ΔU‖;χ_{m-i})<0.95**	**Subspace selection**	**Forward selection**	**Forward selection + whitening**
	100%			
Step 1	8 (100 %)	8	8	8
Step 2	8, 20 (99.9979%)	8, 14	8, 20	8, 14
Step 3	8, 20, 14 (5.37%)	8, 14, 18	8, 20, 17	8, 14, 20

In Scenario 2, the structure sustained damage in elements 11 and 20, with damage severities of 20% each. Applying the SSWT method, the algorithm successfully identified element 11 as the primary damage location in the first step, as indicated by the high probability (98.5%) that the observed changes are not solely due to noise (Table 3). In the subsequent step, with the inclusion of whitening, the algorithm correctly identified element 20, demonstrating the method's ability to pinpoint multiple damage locations. In contrast, both forward selection methods, with and without whitening, also identified element 11 in the first step. However, in the second step, the forward selection without whitening incorrectly identified element 14, highlighting the benefit of the whitening process in improving the accuracy of damage localization. Fig. 14 visually corroborates these findings, illustrating the damage extents detected by the SSWT method, closely aligning with the actual damage locations. The norm of error plot in Fig. 13 further supports the efficacy of Tikhonov regularization in refining the damage quantification

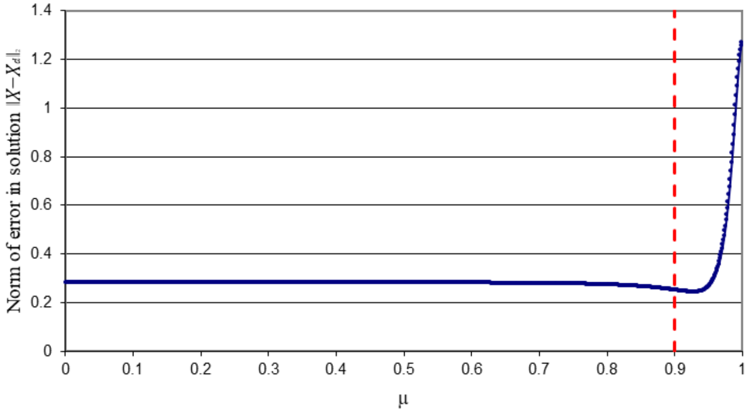


Fig. 13. Norm of error in solution versus $\boldsymbol{\mu}$ for damage scenario 1 and N=0.3%, LE=0.3%

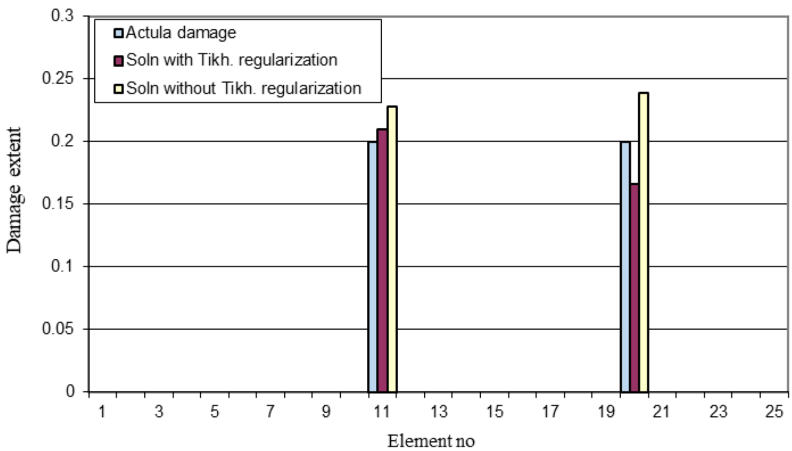


Fig. 14. Damage detection results in the cases of employed and unemployed Tikhonov regularization for damage condition 1

Table 3. Comparison of subspace selection and forward selection in the cases of with and without whitening for damage scenario 2 andN=0.5%, LE=0.5%

Damage scenario 2, N=0.5% LE=0.5% $\frac{\lambda_{max}}{\lambda_{min}}$=31.96	**Subspace selection + whitening CDF(‖ΔU‖;χ_{m-i})<0.95**	**Subspace selection**	**Forward selection**	**Forward selection + whitening**
	100%			
Step 1	11 (98.5 %)	11	11	11
Step 2	11, 20 (21.78%)	11, 14	11, 16	11, 16

Scenario 3 involved damage in elements 2, 9, and 24, with damage severities of 20%, 20%, and 30%, respectively. The SSWT method initiated by correctly identifying element 9 as the most likely damaged element in the first step (Table 4). In the second step, the algorithm, incorporating the whitening process, successfully pinpointed element 24. However, the third damaged element (element 2) was not identified by the SSWT method within the defined stopping criterion. Examining the forward selection methods, it's observed that without whitening, the algorithm identified elements 9 and 17 in the first two steps, with the third step yielding element 18. Conversely, forward selection with whitening identified elements 9 and 19 in the first two steps, and element 8 in the third. This highlights a scenario where while SSWT correctly identifies the initial dominant damages, it might require adjustments to its stopping criteria for accurately detecting all damaged elements in more complex scenarios. The damage detection results are graphically presented in Fig. 15, showing the detected damage in relation to the actual damage. Fig. 16 illustrates the norm of error for this scenario, providing insights into the regularization parameter selection.

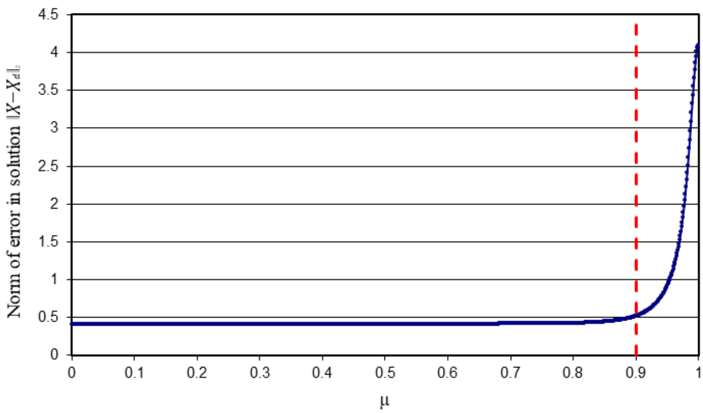


Fig. 15. Norm of error in solution versus $\boldsymbol{\mu}$ for damage scenario 1 and N=0.3%, LE=0.3%

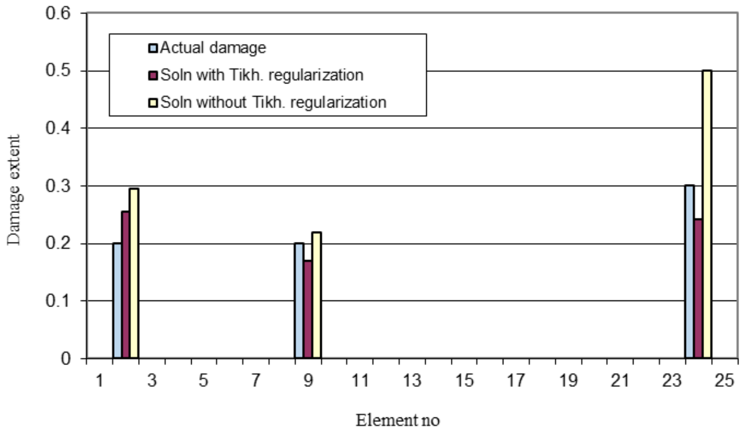


Fig. 16. Damage detection results in the cases of employed and unemployed Tikhonov regularization for damage condition 1

Table 4. Comparison of subspace selection and forward selection in the cases of with and without whitening for damage scenario 3 and N=1%, LE=1%

Damage scenario 3, $N = 1\%$ $LE = 1\%$ $\frac{\lambda_{\max}}{\lambda_{\min}} = 33.47$	**Subspace selection + whitening** $\mathrm{CDF}(\|\Delta\mathrm{U}\|; \chi_{m-i}) < 0.95$	**Subspace selection**	**Forward selection**	**Forward selection + whitening**
	100%			
Step 1	9 (100 %)	9	9	9
Step 2	9, 24 (99.9979%)	9, 17	9, 19	8, 19
Step 3	2, 9, 24 (5.37%)	9, 17, 18	8, 19, 23	8, 19, 20

In Scenario 4, damage was introduced in elements 8, 14, and 19, each with a 20% severity. The SSWT method effectively identified element 8 as the primary damaged element in the initial step (Table 5). Continuing with the whitening process, element 19 was accurately identified in the subsequent step. Similar to Scenario 3, the SSWT method, based on the defined stopping criteria, did not identify the third damaged element (element 14). In comparison, both forward selection approaches successfully identified element 8 in the first step. In the second step, forward selection without whitening identified element 14, while the version with whitening identified element 19. The third step saw forward selection without whitening identify element 18, and forward selection with whitening identify element 14. This again demonstrates the initial effectiveness of SSWT in identifying the most significant damage but suggests potential limitations in detecting all instances of damage within the current stopping rule parameters. The graphical representation of the damage detection results and the norm of error are shown in Figs. 17 and 18, respectively.

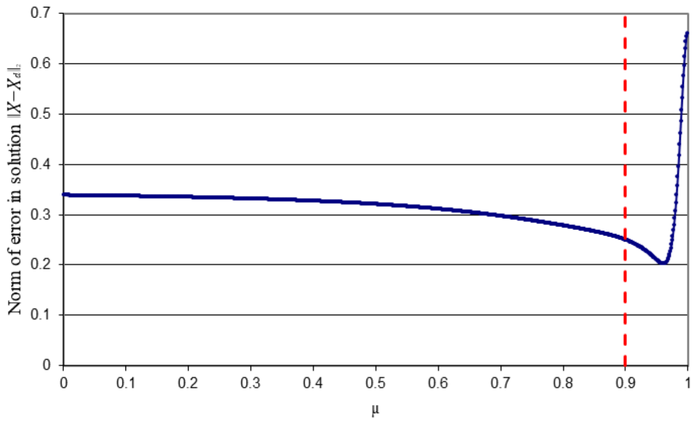


Fig. 17. Norm of error in solution versus $\boldsymbol{\mu}$ for damage scenario 1 and N=0.3%, LE=0.3%

Table 5. Comparison of subspace selection and forward selection in the cases of with and without whitening for damage scenario 4 and N=1%, LE=1%

Damage scenario 4, N=1% LE=1% $\frac{\lambda_{max}}{\lambda_{min}}$=33.80	Subspace selection + whitening CDF($\|\Delta U\|;\chi_{m-i}$)<0.95	Subspace selection	Forward selection	Forward selection + whitening
	100%			
Step 1	8 (100 %)	8	8	8
Step 2	8, 19 (99.01%)	8, 14	8, 19	8, 14
Step 3	8, 19, 14 (5.37%)	8, 14, 18	8, 19, 14	8, 14, 19

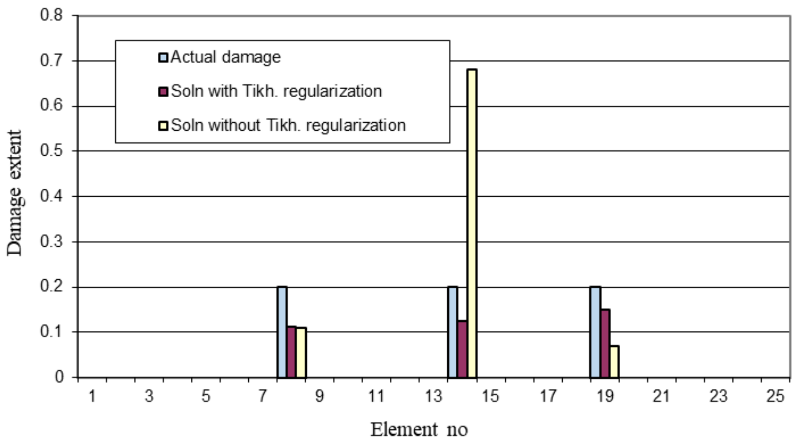


Fig. 18. Damage detection results in the cases of employed and unemployed Tikhonov regularization for damage condition 1

Scenario 5 involved damage in elements 1, 11, and 17, with severities of 25%, 35%, and 20%, respectively. The SSWT method commenced by correctly identifying element 17 as the primary damage location (Table 6). In the subsequent step, the algorithm, with the aid of whitening, accurately detected element 1. However, the third damaged element (element 11) was not identified by SSWT within the stopping criterion. The forward selection method without whitening identified elements 17 and 8 in the first two steps, and element 15 in the third. The forward selection method with whitening identified elements 17 and 1 in the first two steps, and element 14 in the third. This scenario further underscores the SSWT method's initial accuracy in identifying dominant damage but indicates room for improvement in consistently detecting all damaged elements in multi-damage scenarios. The damage detection outcomes and the norm of error plot are visualized in Figs. 19 and 20.

Table 6. Comparison of subspace selection and forward selection in the cases of with and without whitening for damage scenario 5

Damage scenario 5, **N=1.2% LE=1.2%** $\frac{\lambda_{max}}{\lambda_{min}}$**=31.97**	**Subspace selection + whitening CDF(‖ΔU‖;χ_{m-i})<0.95**	**Subspace selection**	**Forward selection**	**Forward selection + whitening**
	99.7 %			
Step 1	17 (98.6 %)	17	17	17
Step 2	1, 17 (97.2 %)	8, 17	17, 18	1, 17
Step 3	1, 11, 17 (5.37%)	8, 17, 15	5, 17, 14	1, 17, 20

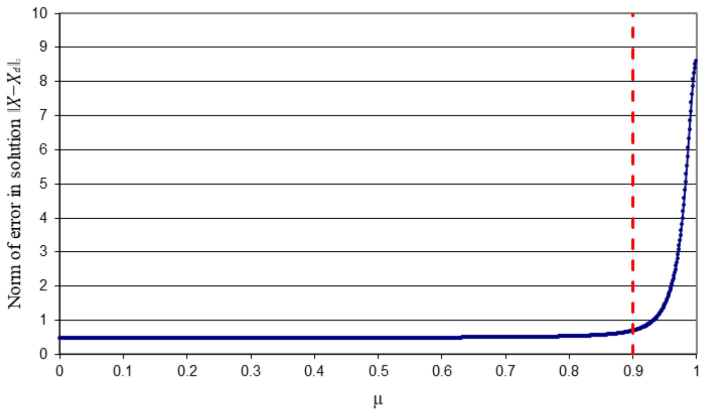


Fig. 19. Norm of error in solution versus $\boldsymbol{\mu}$ for damage scenario 1 and N=0.3%, LE=0.3%

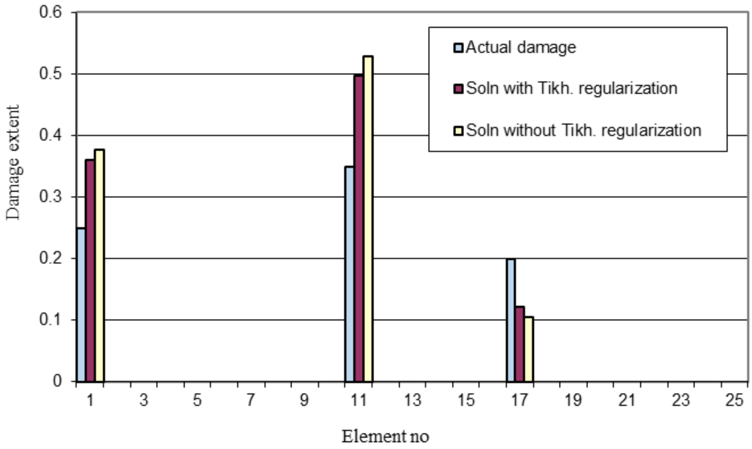


Fig. 20. Damage detection result in the cases of employed and unemployed Tikhonov regularization for damage condition 1

In the final scenario, Scenario 6, the structure experienced damage in elements 3 and 23, with damage severities of 10% and 40%, respectively. The SSWT method successfully identified element 23, the element with the higher damage severity, as the primary damaged element in the first step (Table 7). However, the second damaged element (element 3) was not identified by the SSWT method within the stopping criteria. Examining the forward selection methods, it is noted that without whitening, the algorithm identified elements 21 and 16 in the first two steps. Forward selection with whitening, on the other hand, identified elements 8 and 17 in its first two steps. This scenario reinforces the observation that while SSWT excels at pinpointing the most significant damage, its ability to detect all damaged elements can be influenced by the relative severities of the damage and the chosen stopping criterion. The graphical depiction of the damage detection results and the norm of error are available in Figs. 21 and 22.

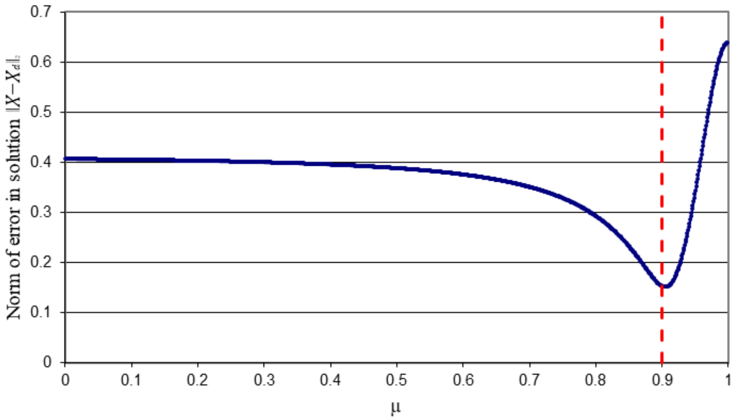


Fig. 21. Norm of error in solution versus $\boldsymbol{\mu}$ for damage scenario 1 and N=0.3%, LE=0.3%

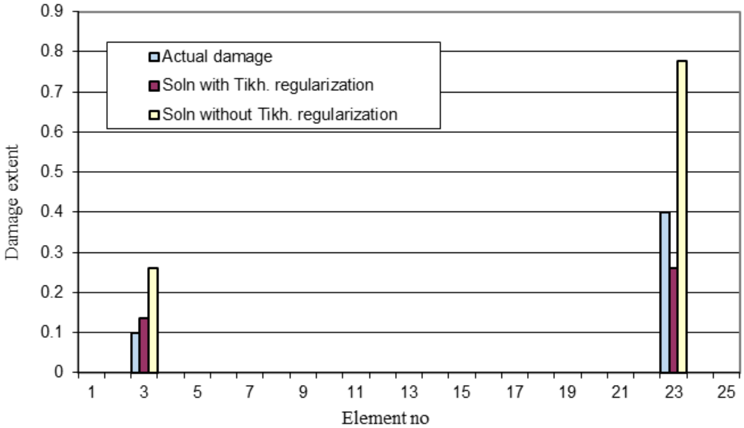


Fig. 22. Damage detection results in the cases of employed and unemployed Tikhonov regularization for damage condition 1

Table 7. Comparison of subspace selection and forward selection in the cases of with and without whitening for damage scenario 6

Damage scenario 6, N=1.5% LE=1.5% $\frac{\lambda_{max}}{\lambda_{min}}$=31.67	**Subspace selection + whitening CDF($\|\Delta U\|$;χ_{m-i})<0.95**	**Subspace selection**	**Forward selection**	**Forward selection + whitening**
	100%			
Step 1	23 (98.7 %)	21	8	23
Step 2	3, 23 (12.7%)	21, 16	8, 17	23, 14

Conclusion

This paper introduced a novel and effective two-stage strategy, termed Subspace Selection with Whitening and Tikhonov Regularization (SSWT), for structural damage detection under static loading conditions, explicitly addressing uncertainties in both applied loads and sensor measurements. A key theoretical contribution was demonstrating the equivalence of combined Gaussian errors in loads and sensors to solely considering Gaussian errors in sensor measurements, provided a specific covariance matrix is utilized. The derivation and calculation of this covariance matrix were explicitly detailed. The proposed SSWT method comprises two distinct stages. The initial stage focuses on identifying potential damage locations by leveraging a whitening transformation. This critical step effectively transforms the ellipsoidal Probability Density Function (PDF) of the equivalent error into a spherical PDF through a linear transformation, thereby converting the error covariance matrix into an identity matrix. This whitening process renews the vector of displacement change and the sensitivity displacement vectors, enabling the selection of the *i* elements corresponding to the subspace that best accommodates the displacement change vector in comparison to subspaces formed by other elements. Recognizing that each subset of elements generates a subspace based on the sensitivity of the displacement vector with respect to the stiffness of those elements, the algorithm iteratively increases *i* from unity until a predefined stopping criterion is met, thus determining the number of damage candidates. The second stage then quantifies the extent of damage in the identified locations using Tikhonov regularization, without requiring iterative updates of the sensitivity matrix.

The efficacy of the proposed SSWT method was rigorously validated through a comprehensive numerical study on a planar truss structure with twenty-five elements, subjected to six distinct damage scenarios and varying levels of measurement and load errors. To demonstrate the proficiency of the initial stage, a direct comparison was made with the well-established forward selection method, both with and without the application of the proposed whitening technique. The results unequivocally demonstrated the superior accuracy of SSWT in detecting damaged members across all six damage scenarios, even in the presence of up to 1.5% error in sensors and applied loads. Specifically, the inclusion of the whitening process led to a significantly more reliable identification of damaged elements compared to scenarios without whitening and when employing the forward selection method. This underscores the crucial role of whitening in mitigating the adverse effects of noise and improving the robustness of the damage localization process. Furthermore, the application of whitening effectively enabled the stopping criterion, based on the null hypothesis regarding the sufficiency of identified damage candidates, to function correctly in all six scenarios, accurately ceasing the search after identifying the appropriate number of damaged elements. Finally, the application of Tikhonov regularization in the second stage yielded promising results in reducing the impact of noise on damage quantification, with the proposed heuristic for selecting the regularization parameter demonstrating its effectiveness. These findings highlight the potential of the SSWT method as a robust and accurate tool for structural health monitoring under realistic operating conditions. Future research directions could explore adaptive strategies for determining the optimal stopping criterion and further refine the selection of the regularization parameter to enhance the method's performance in more complex damage scenarios.

Appendix A: Basic definitions and concepts

Let v and a_i; i=1,…,m be vectors in $R^n (n > m)$ and consider $\mathbf{A} = [a_1, a_2, \ldots, a_m]$. The span of a_i's is defined as the set of all linear combinations of the a_i's. Mathematically speaking, $S = \{\mathbf{A}x : \mathbf{A}x \in R^n\}$ is equivalent to $\mathrm{span}(a_1, a_2, \ldots, a_m) = \mathrm{span}(\mathbf{A})$. The vector v can be decomposed with respect to vector subspace S into two vectors as $v = \mathrm{proj}_S v + v_{\perp S}$, in which $\mathrm{proj}_S \mathbf{v} \in S$ and $\mathbf{v}_{\perp S} \in S^{\perp}$ where $S^{\perp}$ is complementary subspace of S defined as $S^{\perp} = \{w \in R^n : u^T . w = 0, \forall u \in S\}$. The vector $\mathrm{proj}_S \mathbf{v}$ denoting by $\mathbf{v}_{||S}$ is equal to the projection of v onto S. The relation $\|v_{\perp S}\| = \inf\{\|v-u\| : u \in S\}$ always holds, where $\|.\|$ indicates the Euclidean norm, i.e., the nearest point in S to the vector v is given by the projection of v onto S. The components of v along a_i's are determined by $x = A^+ v$, where the symbol "+" indicates Moor-Penrose pseudo-inverse and $x = (x_1, x_2, \ldots, x_m)^T$, in which x_i is the component of v along a_i. The projection of vector v onto S is equal to $\mathrm{proj}_S \mathbf{v} = \mathbf{A}\mathbf{A}^{+}\mathbf{v}$ [5].

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